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Convergence of the vector QD-algorithm. Zeros of vector orthogonal polynomials

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Abstract: The convergence of the vector QD-algorithm, associated to d meromorphic functions, is established. As a consequence a De Montessus–De Ballore theorem for vector Padé approximants is proved. A short numerical study is done in conclusion.

Keywords: Vector Padé approximants, QD-algorithm.

Vector Padé approximants to a function $\mathbb{F} = (f_1, \ldots, f_d)$ from $\mathbb{C}$ to $\mathbb{C}^d$ have been previously defined [7,8]. They are different from the simultaneous Padé Approximants defined by De Bruin [2]. In particular, the denominators of the approximants of degrees $(r+s-1, r)$ are associated to polynomials P_r^s, which for each fixed s, satisfy a recurrence relation of order $(d+1)$, and so have been called vector orthogonal polynomials. As in the scalar case $(d=1)$, a QD-algorithm can be produced, which links the recurrence relations of $(P_r^s)_r$ and $(P_r^{s+1})_r$.

We study here the convergence of that QD-algorithm. In the case where $\mathbb{F}$ is meromorphic, the poles are found as the inverses of the limits of the sequence $(q_r^s)_{s \geqslant 1}$ for $r = 1, 2, \ldots$. Moreover, it is partly possible to know the component f_α of which each $(\lim_{s \to \infty} q_r^s)^{-1}$ is a pole. As a consequence, if $\{z_1, \ldots, z_r\}$ is the union of the poles of all the f_α in a disc D_σ, then the orthogonal polynomials $(P_r^s)_s$ are proved to converge to $\prod_1^r (z - z_i^{-1})$ when s goes to infinity.

As is shown by the results of the second section, although the logical point of view is nearly the same as in the scalar case, the problems and the results are not all the same. Also the proofs need to use generalized Hankel determinants for which the usual properties are no more true.

After the convergence of the QD-algorithm in Section 2, the consequences concerning the vector orthogonal polynomials are studied in Section 3. Some numerical experiments have been made, and are summed up in Section 4.

Let us now sum up the notations which are kept as close as possible to those of the scalar case [1].

$$\mathbb{F}(z) = \sum_0^\infty \Gamma_i z^i, \quad \Gamma_i \in \mathbb{C}^d.$$

If Γ is the functional from $\mathbb{C}^d[[X]] \to \mathbb{C}^d$, defined by $\Gamma(X^i) = \Gamma_i$, then the relations of vector

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orthogonality (R), which define the family of monic polynomials P_r^s ($s \geqslant 0$, $r \geqslant 0$), can be written as follows:

$$r = nd + k: \quad \begin{cases} \Gamma\left(x^i \cdot P_r^s(x)\right) = 0, & i = s, \dots, s+n-1, \\ \Gamma^{(k)}\left(x^{s+n} \cdot P_r^s(x)\right) = 0, & (\Gamma^{(k)}\text{: coordinates of } \Gamma). \end{cases}$$

Then P_r^s can be represented as a quotient of two determinants as follows:

$$P_r^s(x) = \frac{\begin{vmatrix} \Gamma_s & \dots & \Gamma_{s+r} \\ \vdots & & \vdots \\ \Gamma_{s+n-1} & \dots & \Gamma_{s+n+r-1} \\ \Gamma_{s+n}^{(k)} & \dots & \Gamma_{s+n+r}^{(k)} \\ 1 & \dots & X^r \end{vmatrix}}{\begin{vmatrix} \Gamma_s & \dots & \Gamma_{s+r-1} \\ \vdots & & \vdots \\ \Gamma_{s+n-1} & \dots & \Gamma_{s+n+r-1} \\ \Gamma_{s+n}^{(k)} & \dots & \Gamma_{s+n+r-1}^{(k)} \end{vmatrix}} = \frac{H_r^s(x)}{H_r^s}.$$

H_r^s is a determinant $r \times r$, and each row $(\Gamma_s, \dots)$ represents d scalar rows except the last one $(\Gamma_{s+n}^{(k)}, \dots)$, which represents the k first coordinates of $(\Gamma_{s+n}, \dots)$.

1. Evaluation of H_r^s

The computations of this section are similar to those of [3]. D_σ denotes the disc centered at the origin, of radius σ.

Theorem 1.1. *Let the functions f_α, $\alpha = 1, \dots, d$, be meromorphic in a disc D_σ, holomorphic at the origin. Let E_α be the set of poles of f_α and*

$$E = \bigcup_\alpha E_\alpha = \{z_i\}_{i \geqslant 0}.$$

It is supposed that there exists a number r such that there are r poles in D_σ. (The poles are not ordered, by any way.)

$$i = 1, \dots, r, \qquad |z_i| < \sigma, \qquad u_i = z_i^{-1}.$$

For each α, let $V_\alpha = \inf_{E_\alpha \cap D_\sigma} |u_i|$, and ρ_α be any fixed quantity satisfying $V_\alpha > \rho_\alpha > 1/\sigma$. Finally, let

$$a\colon\ 0 < a = \inf_{\alpha = 1, \dots, d} (\rho_\alpha / V_\alpha) < 1.$$

Each function f_α is supposed to have simple poles. Then it follows:

$$\mathbb{F}(t) = \sum_1^r \frac{R_i}{z_i - t} + G(t) \tag{1}$$

with G holomorphic in D_σ and $R_i \in \mathbb{C}^d$.

$$H_r^s = C_r(u_1 \dots u_r)^s(1 + \mathrm{O}(a^s)). \tag{2}$$

C_r is a constant, independent of s:

$$C_r = \begin{vmatrix} R_1 & \dots & R_r \\ R_1 u_1 & \dots & R_r u_r \\ \vdots & & \vdots \\ R_1 u_1^{n-1} & \dots & R_r u_r^{n-1} \\ R_1^{(k)} u_1^n & \dots & R_r^{(k)} u_r^n \end{vmatrix} \cdot \prod_{i<j}(u_i - u_j) \cdot \prod_{i=1}^n u_i. \tag{3}$$

Proof. For each α, $\alpha = 1, \dots, d$, we can write

$$f_\alpha(t) = \sum_{i=1}^r \frac{R_{\alpha i}}{z_i - t} + g_\alpha(t).$$

g_α holomorphic in D_σ. So the relation (1) follows, with $R_i^{\mathrm{T}} = (R_{1i}, \dots, R_{di})$, and $G(t)^{\mathrm{T}} = (g_1(t), \dots, g_d(t))$. G can be expanded in a Taylor series around zero:

$$g_\alpha(t) = \sum_0^\infty b_{\alpha n} t^n, \qquad B_n = (b_{\alpha n})^t_{\alpha = 1,\dots,d}, \qquad \Gamma_n = \sum_{i=1}^r R_i u_i^{n+1} + B_n.$$

Each term Γ_s is replaced by this expression in the Hankel determinant H_r^s:

$$H_r^s = D_r^s + \hat{D}_r^s.$$

D_r^s represents the sum of all the determinants containing only terms $R_i u_i^n$, and $\hat{D}_r^s$ is the sum of determinants containing at least one column $(B_p, \dots, B_{p+n-1}, B_{p+n}^{(k)})^t$.

$$D_r^s = \sum_{i_1,\dots,i_r} \begin{vmatrix} R_{i_1} u_{i_1}^{s+1} & \dots & R_{i_r} u_{i_r}^{s+r+1} \\ \vdots & & \vdots \\ R_{i_1} u_{i_1}^{s+n} & \dots & R_{i_r} u_{i_r}^{s+r+n} \\ R_{i_1}^{(k)} u_{i_1}^{s+n+1} & \dots & R_{i_r}^{(k)} u_{i_r}^{s+r+n+1} \end{vmatrix}$$

$$= u_1^{s+1} \cdots u_r^{s+1} \begin{vmatrix} R_1 & \dots & R_r \\ R_1 u_1 & \dots & R_r u_r \\ \vdots & & \vdots \\ R_1 u_1^{n-1} & \dots & R_r u_r^{n-1} \\ R_1^{(k)} u_1^n & \dots & R_r^{(k)} u_r^n \end{vmatrix} \cdot \sum_\tau \epsilon(\tau) u_{\tau(1)}^0 u_{\tau(2)}^1 \cdots u_{\tau(r)}^{r-1}.$$

The last sum is taken for all the permutations τ of $\{1, \dots, r\}$, and so is a Vandermonde determinant:

$$D_r^s = (u_1 \dots u_r)^{s+1} \begin{vmatrix} R_1 & \dots & R_r \\ \vdots & & \vdots \\ R_1 u_1^{n-1} & \dots & R_r u_r^{n-1} \\ R_1^{(k)} u_1^n & \dots & R_r^{(k)} u_r^n \end{vmatrix} \cdot \begin{vmatrix} 1 & \dots & 1 \\ u_1 & \dots & u_r \\ \vdots & & \vdots \\ u_1^{r-1} & \dots & u_r^{r-1} \end{vmatrix}.$$

And finally, D_r^s verifies:

$$D_r^s = C_r(u_1 \cdots u_r)^s \quad \text{with } C_r \text{ defined by (3).}$$

From Cauchy's estimates applied to the functions g_α, there exists a constant μ such that:

$$|b_{n\alpha}| < \mu\rho_\alpha^n, \quad \alpha = 1, \ldots, d.$$

The sum $\hat{D}_r^s$ is finite, and so is of the same order of its greatest term. So finally,

$$H_r^s = C_r(u_1 \cdots u_r)^s(1 + \mathrm{O}(a^s)). \qquad \square$$

The result can be extended to the case where the poles of f_α are not simple, by a method of confluence as is done in [3].

The preceding result is of interest only if the constant C_r is not zero, and this condition becomes obviously a sufficient condition for the existence of the vector Padé approximants.

The condition "C_r nonzero" will be shown to be equivalent to a case of polewise independent functions $f_1, \ldots, f_d$. This notion has been defined by Graves-Morris and Saff [5], and is recalled now.

Definition. Let each of the functions $f_1, \ldots, f_d$ be meromorphic in the disc D, and let nonnegative integers $\rho_1, \ldots, \rho_d$ be given, for which

$$\sum_{i=1}^{d} \rho_i > 0.$$

Then the functions f_i are said to be polewise independent with respect to the numbers ρ_i, if there do not exist polynomials $\Pi_1, \ldots, \Pi_d$, at least one of which is nonnull (satisfying $\partial^0 \Pi_i \leqslant \rho_i - 1$ if $\rho_i \geqslant 1$ and $\Pi_i \equiv 0$ if $\rho_i = 0$) and such that

$$\psi(z) = \sum_{i=1}^{d} \Pi_i(z) f_i(z)$$

is analytic throughout D.

We get the two following results.

Proposition 1.2. *Let $r = nd + k$, and let a function f_α have less than n poles (in case $1 \leqslant \alpha \leqslant k$), or less than $n - 1$ poles (in case $k + 1 \leqslant \alpha \leqslant d$). Then $C_r = 0$.*

Proof. For fixed α, let us suppose that f_α has $z_1, \ldots, z_p$ as poles. The determinant in C_r, can be expanded in minors of dimension p, and $R_{\alpha i} = 0$ for $i > p$.

Then it is clear that $C_r = 0$ if $p < n + 1$ (for $\alpha \leqslant k$). $\square$

Proposition 1.3. *The condition $C_r \neq 0$ is equivalent to the following: the functions $f_1, \ldots, f_d$ are polewise independent, in the disc D containing r poles, with respect to the integers $(n + \epsilon_\alpha)$, where $\epsilon_\alpha = 1$ if $1 \leqslant \alpha \leqslant k$, $\epsilon_\alpha = 0$ if $k + 1 \leqslant \alpha \leqslant d$.*

Proof. Each function f_α can be written as

$$f_\alpha(z) = \sum_{i=1}^{r} \frac{R_{\alpha i}}{z_i - z} + g_\alpha(z),$$

$$\psi(z) = \sum_{\alpha=1}^{d} \Pi_\alpha(z) \cdot f_\alpha(z) = \sum_{i=1}^{r} \frac{\sum_{\alpha=1}^{d} \Pi_\alpha(z) \cdot R_{\alpha i}}{z_i - z} + \sum_{\alpha=1}^{d} \Pi_\alpha(z) \cdot g_\alpha(z);$$

and ψ analytic is equivalent to the system

$$(\mathrm{S}') \qquad \sum_{\alpha=1}^{d} \Pi_\alpha(z_i) \cdot R_{\alpha i} = 0, \quad i = 1, \ldots, r.$$

It is a linear system of equations with as unknowns the coefficients of the polynomials Π_α. There are $\Sigma\rho_i$ unknowns.

In the case where $\Sigma\rho_i = r$, the existence of the polynomials Π_α, with at least one of which nonnull, is now equivalent to the fact that the determinant Δ' of this system (S′) is zero. As $\Delta' = KC_r$, with K a constant, the proposition follows. □

In the case where $\mathbb{F}$ is a rational function, two results, similar to the scalar case, can be proved.

Theorem 1.4. *If $\mathbb{F}$ is a rational function of type $(r+h,\ r)$, then*

$$\begin{cases} H_m^s = 0, & m > r, \qquad s \geqslant 0; \\ H_r^s = C_r(u_1 \cdots u_r)^s \neq 0, & s > h. \end{cases}$$

The proof is a direct consequence of the preceding result.

Theorem 1.5. *If $H_m^s = 0$ $(m > r,\ s \geqslant 0)$ and $H_r^s \neq 0$ $(s > h)$, then $\mathbb{F}$ is a rational function of type $(r+h,\ r)$.*

Proof. We have to prove that there exists a polynomial of degree r, such that the product

$$\left(\sum_0^r \Gamma_i t^i\right) \cdot \left(\sum_0^r b_j t^j\right)$$

is a vector-polynomial of degree $r+h$.

$$\sum_{i=0}^{r} b_i \Gamma_{r+s-i} = 0, \quad s > h.$$

The first r equations form the following system $(r = nd + k)$, whose determinant is H_r^{h+1}

$$\begin{cases} \displaystyle\sum_{i=0}^{r} b_i \Gamma_{r+s-i} = 0, \quad s = h+1, \ldots, h+n, \\ \displaystyle\sum_{i=0}^{r} b_i \Gamma_{r+n+h+1-i}^{(k)} = 0. \end{cases}$$

There are $(r+1)$ unknowns and this system has a nontrivial solution $(b_0, \ldots, b_r)$, because H_r^{h+1} is nonzero.

The equations which follow for $s \geqslant h+n$ are satisfied, because they are linear combinations of the first ones, H_m^s being zero for $m > r$. □

2. Convergence of the QD-algorithm

The QD-algorithm has been defined for the vectorial case [6,8] by the following relations:

$$\begin{cases} P_{r+1}^s(x) = xP_r^{s+1}(x) - q_{r+1}^s P_r^s(x), & r \geqslant 0, \quad s \geqslant 0, \\ P_r^{s+1}(x) - P_r^s(x) = -\sum\limits_{i=r-d}^{r-1} e_{r,i}^s P_i^{s+1}(x), & r \geqslant d, \quad s \geqslant 0; \end{cases}$$

$$q_{r+1}^s = H_{r+1}^{s+1} H_r^s / H_{r+1}^s H_r^{s+1}.$$

The computation of q_{r+1}^s involves the evaluation of the generalized Hankel determinants of order r and $r+1$: H_r^s, H_r^{s+1}, H_{r+1}^s, H_{r+1}^{s+1}. The hypothesis of Theorem 1.1 taken at the orders r and $r+1$ will lead to limit the algorithm to cases where the sequence $|u_i|$ is decreasing, u_{hd+k} being the inverse of a pole of f_k. This is too restrictive, and the following situation will be studied.

Let $E^\alpha = \{z_i^\alpha\}_{i \geqslant 0}$ be the set of poles of f_α $(\alpha = 1, \ldots, d)$, which can be finite or not; $f_1, \ldots, f_d$ will be said to be r-polewise independents if they are polewise independent with respect to the integers $(n + \epsilon_\alpha)_\alpha$ defined by

$$\text{if } r = nd + k: \begin{cases} \epsilon_\alpha = 1, & 1 \leqslant \alpha \leqslant k, \\ \epsilon_\alpha = 0, & k < \alpha \leqslant d. \end{cases}$$

Theorem 2.1. *We assume that for each* $\alpha = 1, \ldots, d$,

$$|z_1^\alpha| < \cdots < |z_i^\alpha| < \cdots .$$

Each function f has simple poles. $\bigcup_\alpha E_\alpha$ *contains at least m poles in* D_ρ. $f_1, \ldots, f_d$ *are r-polewise independent for* $r = 1, \ldots, m$ *on* D_ρ.

Then, for $r = 1, \ldots, m$

$$\begin{cases} \lim\limits_{s \to \infty} q_r^s = u_r, \\ u_r^{-1} \in \bigcup\limits_\alpha E_\alpha. \end{cases}$$

Proof. For each α, there exists an integer r_α, such that

$$|z_{r_\alpha}^\alpha| < \rho < |z_{r_\alpha+1}^\alpha| \quad \text{and} \quad \sum_\alpha r_\alpha \geqslant r,$$

and the moments c_n^α of f_α, coordinates of Γ_n, are

$$c_n^\alpha = \sum_{r_\alpha}^{i=1} R_{\alpha i} (u_i^\alpha)^{n+1} + b_{\alpha n}, \qquad u_i^\alpha = (z_i^\alpha)^{-1}.$$

Let us prove, by induction on r, that the sequences $(q_r^s)_s$ converge.

$$r=1: \quad H_1^s = c_s^1 = R_{11}\left(u_1^1\right)^{s+1}\left(1+\mathrm{O}\left(\left|u_{r_1}^1/u_1^1\right|^s\right)\right) + b_{1s}.$$

As in Theorem 1.1 there exists a constant μ_1 such that $|b_{1s}| < \mu_1/\rho^s$.

So it follows

$$k_1 = \inf\left(\left|u_{r_1}^1/u_1^1\right|, (1/\rho)/\left|u_1^1\right|\right) < 1, \qquad H_1^s = R_{11}\left(u_1^1\right)^{s+1}\left(1+\mathrm{O}\left(k_1^s\right)\right),$$

$$\lim_{s\to\infty} q_1^s = \lim_{s\to\infty} c_{s+1}^1/c_s^1 = u_1^1,$$

$$r=2: \quad \text{if } u_1^1 = u_1^2, \quad c_s^2 = R_{21}\left(u_1^1\right)^{s+1} + R_{22}\left(u_2^2\right)^{s+1}\left(1+\mathrm{O}\left(\left|u_{r2}^2/u_2^2\right|^2\right)\right) + b_{2s}.$$

As before: $|b_{2s}| < \mu_2(1/\rho)^s$ and let

$$k_2 = \inf\left(\left|u_{r_1}^2/u_2^2\right|, (1/\rho)/\left|u_2^2\right|\right), \quad k_2 < 1,$$

$$c_s^2 = R_{21}\left(u_1^1\right)^{s+1} + R_{22}\left(u_2^2\right)^{s+1}\left(1+\mathrm{O}\left(k_2^s\right)\right),$$

$$H_2^s = \begin{vmatrix} c_s^1 & c_{s+1}^1 \\ c_s^2 & c_{s+1}^2 \end{vmatrix}$$

$$= R_{11}R_{22}\left(u_1^1\right)^{s+1}\left(u_2^2\right)^{s+1}\left(u_1^1 - u_2^2\right)\left(1+\mathrm{O}\left(k_1^2\right)\right)\left(1+\mathrm{O}\left(k_2^s\right)\right),$$

$$q_2^s = H_2^{s+1}\cdot c_s^1/H_2^s c_{s+1}^1, \qquad \lim_{s\to\infty} q_2^s = u_2^2.$$

If $u_1^1 \neq u_1^2$,

$$c_s^2 = R_{21}\left(u_1^2\right)^{s+1}\left(1+\mathrm{O}\left(k_2^{\prime s}\right)\right)$$

where

$$k_2' = \inf\left(\left|u_{r_2}^2/u_1^2\right|, (1/\rho)/\left|u_1^2\right|\right) < 1,$$

$$H_2^s = R_{11}R_{21}\left(u_1^1\right)^{s+1}\left(u_1^1 - u_1^2\right)\left(1+\mathrm{O}\left(k_1^s\right)\right)\left(1+\mathrm{O}\left(k_2^{\prime s}\right)\right),$$

$$\lim_{s\to\infty} q_2^s = u_1^2.$$

Let us define u_1, u_2:

$$u_1: \ |u_1| = \inf |\{u_i,\ u_i \in E^1\},$$

$$u_2: \ |u_2| = \inf\{\,|u_i|,\ u_i \in E^2 - E^1 \cap E^2\}.$$

In both cases

$$\begin{cases} \lim\limits_{s\to\infty} q_1^s = u_1, \\ \lim\limits_{s\to\infty} q_2^s = u_2. \end{cases}$$

The same thing can be observed when going from the step r to the step $r+1$ ($r+1 \leqslant m$). We assume that

$$i = 1,\ldots,r, \qquad \lim_{s\to\infty} q_i^s = u_i, \quad u_i^{-1} \in \bigcup_\alpha E_\alpha.$$

We note E_r^α the subset of E^α found at the step r. Because $f_1, \dots, f_d$ have been supposed to be r-polewise independent:

$$\text{Card } E_r \geqslant n + \epsilon_\alpha, \quad r = nd + k,$$

if Card $E_r^{k+1} = n$, E_{r+1}^{k+1} must contain $n+1$ elements.

Let u_{r+1} such that

$$u_{r+1} = u_{n+1}^{k+1},$$

$$c_s^{k+1} = \sum_1^{n+1} R_{k+1,i}\left(u_i^{k+1}\right)^{n+1}\left(1 + \mathrm{O}\left(\left|u_{r_{k+1}}^{k+1}/u_{n+1}^{k+1}\right|^s\right) + b_{k+1,n},\right.$$

$$H_{r+1}^s = C_r(u_1 \cdots u_r u_{r+1})^s(1 + \mathrm{O}(k_1^s)) \cdots (1 + \mathrm{O}(k_{r+1}^s)),$$

$$k_{r+1} = \inf\left(\left|u_{r_{k+1}}^{k+1}/u_{n+1}^{k+1}\right|, (1/\rho)/u_{n+1}^{k+1}\right),$$

$$\lim_{s \to \infty} q_{r+1}^s = u_{r+1}, \quad u_{r+1} \in E^{k+1},$$

if Card $E_r^{k+1} \geqslant n+1$, let us call u_{r+1} the following:

$$|u_{r+1}| = \sup\left\{ |u_i|, u_i^{-1} \notin \bigcup_\alpha E_r^\alpha \right\}.$$

It is always possible to find u_{r+1} if $(\bigcup_\alpha E^\alpha - \bigcup_\alpha E_r^\alpha)$ is not empty. But if it was empty, the constant C_r ought to be zero, and that is not possible because $f_1, \dots, f_d$ are $(r+1)$-polewise independent. If $u_{r+1}^{-1} \in E^\alpha$,

$$c_s^\alpha = \sum_{E_r^\alpha} R_{\alpha i}(u_i)^{s+1} + R_{\alpha,r+1}(u_{r+1})^{s+1}(1 + \mathrm{O}(h_\alpha^s)) + b_{\alpha s},$$

$$h_\alpha = \left|u_{r_\alpha}^\alpha / u_{r+1}\right|.$$

If $u_{r+1}^{-1} \notin E^\alpha$, let u_*: $|u|_* = \inf\{ |u_i|, u_i \in E_r^\alpha\}$,

$$c_s^\alpha = \sum_{E_r^\alpha - \{u\}} R_{\alpha i}(u_i)^{s+1} + R_{\alpha *} \cdot u_*^{s+1}(1 + \mathrm{O}(h_\alpha^s)) + b_{\alpha s},$$

$$h_\alpha = \left|u_{r_\alpha}^\alpha / u_*\right|.$$

Then we get $H_{r+1}^s = C_{r+1}(u_1 \cdots u_r u_{r+1})^s(1 + \mathrm{O}(k_1^s)) \cdots (1 + \mathrm{O}(k_{r+1}^s))$,

$$k_{r+1} = \inf_\alpha\left(h_\alpha, (1/\rho)/\left|u_{r_\alpha}^\alpha\right|\right) < 1, \qquad \lim_{s+\infty} q_{r+1}^s = u_{r+1}. \qquad \square$$

The proof can be extended to the case where f_α has multiple poles as is done by Henrici [3].

At the step $r+1$, it is not possible to know of which function f_α, u_{r+1}^{-1} is a pole. This problem comes from the fact that if a pole z_i is simple for a function but common to several ones, it will appear only once in the sequence formed by the limits $(\lim_{s \to \infty}(q_r^s)^{-1})_r$.

From the knowledge of the sequence $(\lim_{s \to \infty}(q_r^s)^{-1})_r$, it is possible to build two sets F^α and G^α

$$F^\alpha \subset E^\alpha \subset G^\alpha.$$

The points in F^α are really poles (with residue nonzero) of f_α, and the points of G^α can be poles of f_α.

We sum up the results at each step, on two examples, according to the facts that for each α the sequence $(|u_i^\alpha|^{-1})$ is increasing, and that Card $E_r^\alpha \geqslant n + \epsilon_\alpha$. We write the points of G_α, and underline the points of F^α. The two examples below, are not real ones. It is what would have been found from the indicated functions.

Example 1.

$$d = 2, \qquad \left(\lim_{s\to\infty} (q_r^s)^{-1}\right)_{r=1,4} = (0.8, 0.6, 0.9, 0.7),$$

$$f_1(z) = \frac{1}{(z-0.8)(z-0.9)}, \qquad f_2(z) = \frac{1}{(z-0.6)(z-0.7)}.$$

	E^1	E^2
$r=1$	$\underline{0.8}$	
$r=2$	$\underline{0.8}$	$\underline{0.6}$
$r=3$	$\underline{0.8}, \underline{0.9}$	$\underline{0.6}, 0.9$
$r=4$	$\underline{0.8}, \underline{0.9}$	$\underline{0.6}, \underline{0.7}$

Example 2.

$$d = 3, \qquad \left(\lim_{s\to\infty} (q_r^s)^{-1}\right)_{r=1,6} = (1.1, 1.0, 1.2, 1.15, 1.05, 1.25),$$

$$f_1(z) = \frac{1}{(z-1.1)(z-1.5)}, \qquad f_2(z) = \frac{1}{(z-1)(z-1.05)},$$

$$f_3(z) = \frac{1}{(z-1.2)(z-1.25)}.$$

	E^1	E^2	E^3
$r=1$	$\underline{1.1}$		
$r=2$	$\underline{1.1}$	$\underline{1.0}$	
$r=3$	$\underline{1.1}, 1.2$	$\underline{1}, 1.2$	$\underline{1.2}$
$r=4$	$\underline{1.1}, \underline{1.15}$	$\underline{1}$, 1.2 or 1.15	$\underline{1.2}$
$r=5$	$\underline{1.1}, \underline{1.15}$	$\underline{1}, \underline{1.05}$	$\underline{1.2}$
$r=6$	$\underline{1.1}, \underline{1.15}, 1.25$	$\underline{1}, \underline{1.05}, 1.25$	$\underline{1.2}, \underline{1.25}$

If the sequence $(z_i)_{i=1,\ldots,r}$ $(z_i = \lim_{s\to\infty}(q_i^s)^{-1})$, is increasing in modulus, we only know:

$$E_r^\alpha \subset \{z_1, \ldots, z_r\}, \quad \alpha = 1, \ldots, d.$$

The sets F^α and G^α depend on the order put on the $\alpha = 1, \ldots, d$.

Let us now look at the sequences $(e_{r,i}^s)_s$. For each r and s, there are d terms $e_{r,i}^s$ instead of one in the scalar case

$$P_r^{s+1}(x) - P_r^s(x) = -\sum_{i=r-d}^{r-1} e_{r,i}^s P_i^{s+1}(x), \qquad r \geqslant d, \quad s \geqslant 0.$$

The $e^s_{r,i}$, written λ_i, are the solution of the following linear system (4), where the functionals C^i are the coordinates of $\Gamma = (C^1, \ldots, C^d)$

$$\begin{cases} \lambda_{r-d} C^{k+1}\left(x^{s+n} P^{s+1}_{r-d}\right) & = C^{k+1}\left(x^{s+n} P^s_r\right), \\ \vdots & \\ \lambda_{r-d} C^{d}\left(x^{s+n} P^{s+1}_{r-d}\right) + \cdots + \lambda_{r-k-1} C^{d}\left(x^{s+n} P^{s+1}_{r-k-1}\right) & = C^{d}\left(x^{s+n} P^s_r\right), \\ \lambda_{r-d} C^{1}\left(x^{s+n+1} P^{s+1}_{r-d}\right) + \cdots + \lambda_{r-k} C^{1}\left(x^{s+n+1} P^{s+1}_{r-k}\right) & = C^{1}\left(x^{s+n+1} P^s_r\right), \\ \vdots & \\ \lambda_{r-d} C^{k}\left(x^{s+n+1} P^{s+1}_{r-d}\right) + \cdots + \lambda_{r-1} C^{k}\left(x^{s+n+1} P^{s+1}_{r-1}\right) & = C^{k}\left(x^{s+n+1} P^s_r\right). \end{cases} \tag{4}$$

The polynomials $P^{s+1}_{r-d}, \ldots, P^{s+1}_r$, P^s_r are supposed to exist, and so the corresponding generalized Hankel determinants H^{s+1}_{r-d+i} $(i = 0, \ldots, d)$, H^s_r, are nonzero. The following notation is now introduced to study $(e^s_{r,i})_s$ for $i = r-d, \ldots, r-1$. $H^s_{r,k+i}$ is defined for $i = 1, \ldots, d$ by modifying the last row of H^s_{r+1} as

$$\begin{array}{ll} \left(c^{k+i}_{s+n} \cdots c^{k+i}_{s+n+r}\right) & \text{if } k+i \leqslant d, \\ \left(c^{k+i-d}_{s+n+1} \cdots c^{k+i-d}_{s+n+1+r}\right) & \text{if } d < k+i. \end{array}$$

Proposition 2.2. *If* $\mathbb{F}$ *is a rational function, of total polar order* r, *then*

$$\begin{array}{lll} H^s_{k,k+i} = 0, & i = 1, \ldots, d, & s \geqslant 0, \\ e^s_{r,i} = 0, & i = r-d, \ldots, r-1, & s \geqslant 0. \end{array}$$

Proof. Following the same method as in Theorem 1.1, it is easy to see that, for $i \geqslant 1$

$$H^s_{r,k+1} = 0, \quad s \geqslant 0.$$

So, the linear system (4) has zero as right member. The determinant of the system is

$$\prod_{i=0}^{d-1} H^{s+1}_{r-d+i+1} / H^{s+1}_{r-d+i} \neq 0.$$

So, the solution $(e^s_{r,i})_{i=r-d,\ldots,r-1}$ is unique and

$$e^s_{r,i} = 0, \quad i = r-d, \ldots, r-1. \qquad \square$$

Proposition 2.3. *With the same assumption as for Theorem* 2.1, *there exist constants* K_r, $K_{r,i}$ such that

$$e^s_{r,r-d} \cong \frac{K_r}{u_{r-d+1}} \left(\frac{u_{r+1}}{u_{r-d+1}}\right)^s, \qquad e^s_{r,r-d+i} \cong \frac{K_{r,i}}{u_{r-d+i+1}} \left(\frac{u_{r+1}}{u_{r-d+i+1}}\right)^s.$$

Proof. As in Theorem 1.1, an evaluation of $H^s_{r,k+i}$ can be found, with constants $C_{r,k+i}$ and a_i

$$H^s_{r,k+i} = C_{r,k+i} \left(u_1 \cdots u_{r+1}\right)^s \left(1 + \mathrm{O}\left(a_i^s\right)\right), \quad a_i < 1.$$

The system which defines the $e^s_{r,i}$ can be written with $H^{s+1}_{r-d+i,k+h}/H^{s+1}_{r-d+i}$ as coefficients, and $H^s_{r,k+h}/H^s_r$ as second member, in the equation h. Then using Cramer's formulas,

$$e^s_{r,r-d+i} = \frac{H^{s+1}_{r-d+1}}{H^{s+1}_{r-d+i+1}} \begin{vmatrix} H^{s+1}_{r-d+1} & & & H^s_{r+1} \\ \vdots & \ddots & & \vdots \\ & & H^{s+1}_{r-d+i} & H^s_{r,k+i} \\ H^{s+1}_{r-d,k+i+1} & \cdots & H_{r-d+i-1,k+i+1} & H^s_{r,k+i+1} \end{vmatrix}$$

$$\times \frac{1}{H^{s+1}_{r-d+1} \cdots H^{s+1}_{r-d+i} \cdot H^s_r},$$

$$\lim_{s\to\infty} e^s_{r,r-d+i} = \frac{C_{r-d+i}}{C_{r-d+i+1}} \begin{vmatrix} C_{r-d+1} & & & C_{r+1} \\ \vdots & \ddots & & \vdots \\ & & C_{r-d+i} & C_{r,k+i} \\ C_{r-d,k+i+1} & \cdots & C_{r-d+i-1,k+i+1} & C_{r,k+i+1} \end{vmatrix}$$

$$\times \frac{1}{C_{r-d+1} \cdots C_{r-d+i} \cdot C_r (u_{r-d+i+1})^{s+1}}.$$

The result follows

$$i = 0, \ldots, d-1, \quad \lim_{s\to\infty} e^s_{r,r-d+i} = K_{r,i} \frac{(u_{r+1})^s}{(u_{r-d+i+1})^{s+1}}. \qquad \square$$

As it has been shown, u_{r-d+h} and u_{r+h} do not necessarily correspond to poles of the same function f_α. So although the poles of each f_α are supposed to form a strictly increasing sequence (in modulus), the limit of $e^s_{r,i}$ is not necessarily zero and so this result does not have the same interest as the analogous one for the scalar case.

The rate of convergence of the sequence $(q^s_r)_s$ can be studied [4], if d is odd.

The sequence

$$t_s(r) = H^{s+1}_r / H^s_r$$

is linear, and converges to $(u_1 \cdots u_r)$.

Then, as

$$q^s_{r+1} = t_s(r+1)/t_s(r).$$

$(q^s_{r+1})_s$ is linear if the speed of convergence of $(t_s(r+1))_s$ and that of $(t_s(r))_s$ have a different modulus

$$|u_{r+1}/u_r| \neq |u_{r+2}/u_{r+1}|.$$

The proofs, for the case d odd, can be found in [8]. They are long, mainly because the generalized Hankel determinants do not verify the classical identity

$$H^{s+2}_r H^s_r - \left(H^{s+1}_r\right)^2 = H^{s+2}_{r+1} \cdot H^s_{r-1}.$$

The fact that the convergence of $(q^s_r)_s$ is linear remains to be proved in the case where d is even but, as it can be seen from the numerical study in Section 4, it seems to be a problem of identities of determinants, and the result is probably true for any d.

3. Zeros of vector-orthogonal polynomials

Theorem 3.1. *Under the same hypothesis as Theorem* 2.1, *we get*

$$H_r^s(x) = C_r(u_1 \cdots u_r)^s \left[\prod_1^r (x - u_i) + \mathrm{O}(a^s) \right], \quad a < 1.$$

This relations holds uniformly, for x in a bounded set.

Proof.

$$H_r^s(x) = \begin{vmatrix} \Gamma_s & \cdots & \Gamma_{s+r} \\ \vdots & & \vdots \\ \Gamma_{s+n-1} & \cdots & \Gamma_{s+r+n-1} \\ \Gamma_{s+n}^{(k)} & \cdots & \Gamma_{s+r+n}^{(k)} \\ 1 & \cdots & x^r \end{vmatrix} = D_r^s(x) + \hat{D}_r^s(x),$$

$$\Gamma_s = \sum_{i=1}^n R_i u_i^{s+1} + B_s.$$

$D_r^s(x)$ represents the sum of all the determinants containing at least one "vector" row of type $(B_h, \ldots, B_{h+r})$, a vector row is the set of d rows.

Developing now D_r^s still according to the rows, we get

$$D_r^s(x) = \sum_{i_1, \ldots, i_r} \begin{vmatrix} R_{i_1} u_{i_1}^{s+1} & \cdots & R_{i_1} u_{i_1}^{s+r+1} \\ \vdots & & \vdots \\ R_{i_{r-1}} u_{i_{r-1}}^{s+n} & \cdots & R_{i_{r-1}} u_{i_{r-1}}^{s+n+r} \\ R_{i_r}^{(k)} u_{i_r}^{s+n+1} & \cdots & R_{i_r}^{(k)} u_{i_r}^{s+r+n+1} \\ 1 & \cdots & x^r \end{vmatrix}.$$

It suffices now to notice that $D_r^s(x)$ is a polynomial of degree r, with leading coefficient $C_r(u_1 \cdots u_r)^s$, and which verifies

$$D_r^s(u_i) = 0, \quad i = 1, \ldots, r.$$

It follows that

$$D_r^s(x) = C_r(u_1 \cdots u_r)^s \prod_{i=1}^r (x - u_i).$$

For the determinant $\hat{D}_r^s(x)$, if x is in a bounded set, the same proof as in Theorem 1.1, leads to:

$$\hat{D}_r^s(x) = (u_1 \cdots u_r)^s \mathrm{O}(a^s), \quad a < 1.$$

Finally, we get the result

$$H_r^s(x) = C_r(u_1 \cdots u_r)^s \left(\prod_{i=1}^{r} (x - u_i) + \mathrm{O}(a^s) \right). \qquad \square$$

The result can be extended to the case where the poles of f_α are not simple.

Corollary 3.2. *The zeros of P_r^s tend to the u_i as s goes to infinity.*

Corollary 3.3. *If $\mathbb{F}$ is a rational function of type (h, r), the orthogonal polynomials are exactly $P_r^s(x) = \prod_1^r (x - u_i)$, $s \geqslant 0$.*

The result can be considered as a De Montessus–De Ballore theorem for the vector Padé approximants of a meromorphic function $\mathbb{F}$, and so is to be linked with the result of [5] concerning simultaneous Padé approximants.

4. Numerical examples

The computations have been performed on a computer DPS8 multics (simple precision, 8 digits). The double precision has not been used.

The tested functions are rational functions of the form

$$1 / \prod_1^{n_\alpha} (x - z_j^\alpha), \quad \alpha = 1, \ldots, d.$$

The first problem is the initialisation of the algorithm, and then the stability of the algorithm. This second problem must be studied, but the stability does not seem to be worse than in the scalar case. The case where d is even looks better than d odd from this point of view.

Example 1. $d = 2$, $n_1 = 3$, $n_2 = 2$. The poles are (1.3, 1.2, 1.1) and (1.0, 0.9)

z_i^{-1}	$(1.1)^{-1}$ 0.9090	$(0.9)^{-2}$ 1.1111	$(1.2)^{-1}$ 0.83333	$(1)^{-1}$ 1.0	$(1.3)^{-1}$ 0.7679
s	q_1^s	q_2^s	q_3^s	q_4^s	q_5^s
32	0.9180	1.118	0.831	1.004	0.750
64	0.9096	1.1114	0.83338	1.0003	0.7679

In that case $e_r^5 \sim 0$.

Example 2. $d = 2$, $n_1 = 3$, $n_2 = 2$. Common poles to f_1 and f_2 (0.9, 1.0, 1.0) and (0.9, 1)

s	$(0.9)^{-1}$ 1.1111	1.0	1.0
32	1.124	1.034	0.95
48	1.114	1.024	0.971
64	1.1118	1.018	0.980

0.9 appears once because it is a simple pole, and 1 twice because it is a pole of order 2 for f_1.

The convergence is not as good as in the preceeding case.

The computations are more expensive when d increases. To see the effect of this remark, we compare the sequences converging to the second pole of the third function if d equals 3, 4, 5 and 6.

The poles are (1.0, 1.0, 1.0), (1.2, 1.3), (1.4, 1.5), (0.8, 0.9), 1.05, 1.15), (0.85, 0.95) for f_1, f_2, f_3, f_4, f_5, f_6 resp.

The studied sequences are all converging to $(1.5)^{-1} = 0.6666$.

s	$d=3$	$d=4$	$d=5$	$d=6$
	q_6^s	q_7^s	q_8^s	q_9^s
39	0.762	0.655	0.654	0.664
45	0.715	0.655	0.640	0.660
53	0.698	0.655	0.634	0.658

In all the cases, if the sequences are transformed by Aitken's Δ^2 process, the sequences are accelerated, which is a logical consequence of their linear rate of convergence, d being odd, or even.

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